

An overview of the quantum marginal problem

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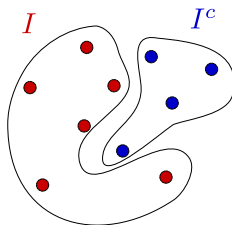
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- 1 What are marginals?
- 2 Pure univariate problem
- 3 Applications
 - Entanglement polytopes
 - Pinning
- 4 Summary

Outline

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Classical vs Quantum



Classical marginals

n random variables, joint distribution

$p(x_1, \dots, x_n)$.

Marginal distribution of subset I is

$$p(I) := \sum_{I^c} p(x_1, \dots, x_n)$$

Quantum marginals

n particle system with state (density matrix) ρ .

Marginal of subset I (reduced density matrix) is

$$\rho_I := \text{tr}_{I^c} \rho$$

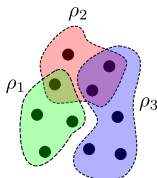
If $|I| = m$, ρ_I is called m -RDM.

Overlapping vs non-overlapping

The question

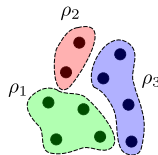
When is a set of marginals $\rho_1, \dots, \rho_r$ compatible with a given global state ρ ?

Overlapping



- Computationally intractable (QMA-Hard)

Non-overlapping



- Possibly tractable
- A solution $\exists$ in this case

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Klyachko's solution

- Klyachko (2004) has come up with a solution (algorithm)
- Relating the trace operation to that of taking the *moment map* (symplectic geometry)
- Important result (convexity of the moment map): The spectra of 1-RDMs is convex polytope! (highly non-trivial)
- Valid for any kind of symmetry (indistinguishable particles, fermions, etc...)

Qubits

1RDM

Consider a pure state $|\psi\rangle$ of n qubits.

$$\rho_i^{(1)} := \text{tr}_{\setminus i} |\psi\rangle \langle \psi| \quad (1\text{RDM}) \quad (1)$$

$$\text{spec} \rho_i^{(1)} = (\lambda_{\min}^{(i)}, \lambda_{\max}^{(i)}) \quad (2)$$

$$\lambda_{\min}^{(i)} + \lambda_{\max}^{(i)} = 1 \quad (\text{Constraint - trace condition}) \quad (3)$$

Higuchi inequalities [Higuchi et al. (2003)]

$$\lambda_{\max}^{(i)} \geq \sum_{i \neq j} \lambda_{\max}^{(j)} - (n-2), i = 1, \dots, n$$

$$(\lambda_{\max}^{(i)} \in [0.5, 1])$$

(4)

Necessary and sufficient

3 qubit Higuchi polytope

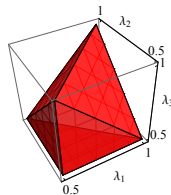


Figure 1: Higuchi polytope - a convex polytope.

Fermions: Generalized Pauli constraints

1 RDM

- $|\psi\rangle \in \mathcal{H} = \Lambda^N(\mathbb{C}^k)$, i.e., N fermions, k modes.
- Indistinguishability $\Rightarrow$ only one 1RDM.

$$\begin{aligned}\rho_{i,j}^{(i)} &= \langle \psi | a_i^\dagger a_j | \psi \rangle \\ \text{spec } \rho &= (\lambda_1, \dots, \lambda_k) \\ \lambda_i &\geq \lambda_{i+1}\end{aligned}\tag{5}$$

Pauli principle

$$\lambda_i \in [0, 1]\tag{6}$$

Generalized Pauli constraints

Affine inequalities of the form

$$D(\lambda_1, \dots, \lambda_k) = c_0 + \sum_{i=1}^k c_i \lambda_i \geq 0\tag{7}$$

Fermions: Dennis-Borland setting

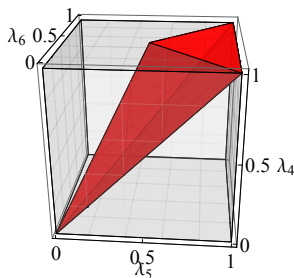


Figure 2: Dennis-Borland (red) and Pauli (gray) polytopes.

Dennis-Borland setting

$\mathcal{H} = \Lambda^3(\mathbb{C}^6)$, i.e., 3 fermions, 6 modes

$$\lambda_1 + \lambda_6 = \lambda_2 + \lambda_5 = \lambda_3 + \lambda_4 = 1$$

$$\lambda_5 + \lambda_6 - \lambda_4 \leq 0$$

(8)

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SLOCC classes

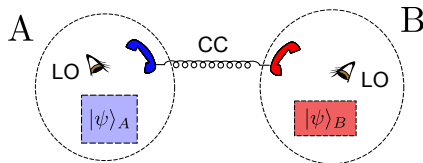


Figure 3: Example of LOCC operations on a global state $|\psi\rangle$ with reduction $|\psi\rangle_A$ and $|\psi\rangle_B$.

Entanglement Classes

Two pure states $|\psi\rangle, |\phi\rangle$ are in the same entanglement class if they can be converted into each other with finite prob. using LOCC. This is called SLOCC.

Mathematically:

$$|\psi\rangle \sim |\phi\rangle \Leftrightarrow |\psi\rangle = M_1 \otimes \dots \otimes M_N |\phi\rangle \quad (9)$$

$$M_i \in SL(\mathbb{C}^2).$$

2,3-partite entanglement

Bi-partite

- Separable states: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$
- Entangled states: $|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$

Tri-partite

- Separable states: $|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle$
- 3 classes of bi-separable states: $|\psi_1\rangle \otimes |\psi_{2,3}\rangle, \dots$
- W class: $|W\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle)$
- GHZ class: $|GHZ\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$

Multi-partite entanglement

The problem

- At least $2^{n+1} - 6n - 2$ real parameters to parametrize classes
- Each parameter is experimentally difficult to access

A solution (for sufficiently pure states)

Turns out that looking at local information only:

- Provides rich information about global entanglement
- Scales linearly with n
- Is easy to access experimentally

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Entanglement polytopes

Entanglement polytopes [M. Walter et al. (2012)]

- To every SLOCC class $\mathcal{C} \rightarrow$ associated set $\Delta_{\mathcal{C}}$ of local spectra $\in \mathcal{C}$
- Turns out $\Delta_{\mathcal{C}}$ is again a convex polytope: *entanglement polytope*

Properties

- Efficient in the number of parameters (requires only local parameters)
- For any n there are only finitely many entanglement polytopes (unlike SLOCC classes)

3-partite polytopes

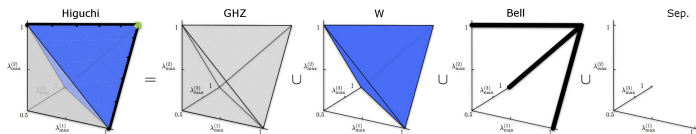
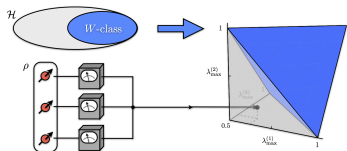


Figure 4: Higgs polytope and the entanglement classes polytopes for a tripartite system (Walter et al. (2013)).



W-class corresponds to upper pyramid.
Any violation witnesses GHZ-type entanglement.

Figure 5: Witnessing GHZ-type entanglement (Walter et al. (2013)).

Pinning - Cursory interview

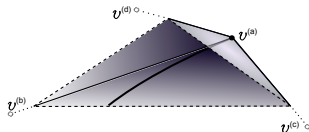


Figure 6: Trajectory of a 1RDM spectrum of ground state. It can't leave the polytope Schilling et al. (2013).

- We know states on polytope boundary have a *simple form*
- *Hypothesis (Klyachko)*: Ground state may be prevented from being further minimized by polytope boundary
- Schilling et al. (2013) showed that Harmonium displays *quasi-pinning* and not *exact pinning*
- Q.: Can we make use of this to reduce Hilbert space where to look for GS?

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Summary

- Presented quantum marginals
- Some computational complexity results on the quantum marginal problem
- $\exists$ solution to the pure univariate (1-body) quantum marginal problem
- Showed some applications of marginals

What we are really doing

- Trying to translate the results into an easier language
- Coming up with scenarios where one can get closed form solutions
- Trying to understand the real implications to physics/chemistry and finding further applications of marginals